Research prospectus

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1 Motivation

My research interests lie in the area of the moduli of Riemann surfaces and its interplay with other fields of mathematics, especially hyperbolic geometry, algebraic geometry, symplectic geometry and ergodic theory.

Let X be a complete hyperbolic Riemann surface of genus g with n punctures. My work has been motivated by the problem of estimating $s_X(L)$, the number of primitive *simple* closed geodesics of hyperbolic length less than L on X. To explore this problem, we have followed two approaches: the first using symplectic geometry of moduli spaces of curves, and the second using ergodic theory of the earthquake flow. Both methods provide new results and insights about the moduli space $\mathcal{M}_{g,n}(\ell_1, \ldots, \ell_n)$ of Riemann surfaces with geodesic boundary components, the bundle of holomorphic quadratic differentials over $\mathcal{M}_{g,n}$ and the space of measured laminations.

Our main results are the following.

- We give a recursive method for calculating the Weil-Petersson volume $V_{g,n}(\ell)$ of the moduli space $\mathcal{M}_{g,n}(\ell)$ of bordered Riemann surfaces with fixed boundary lengths and show that $V_{g,n}(\ell)$ is a polynomial in $\ell_1, \ldots, \ell_n$ (§2).

- We give a new proof of the Witten-Kontsevich formula for the intersection numbers of tautological classes on $\overline{\mathcal{M}}_{g,n}$ using hyperbolic geometry (§3).

- We show that the number of simple closed geodesics of length $\leq L$ on $X \in \mathcal{M}_{g,n}$ has the asymptotic behavior

  $$s_X(L) \sim n_X L^{6g-6+2n}$$

  as $L \to \infty$ (§4, §6).

- We establish a relationship between the earthquake flow and the Teichmüller horocycle flow, leading to the proof of the ergodicity of the earthquake flow on moduli space (§5).

- We calculate the volume of the moduli space $\mathcal{Q}^1\mathcal{M}_{g,n}$ of unit-norm holomorphic quadratic differentials. Remarkably, the answer is given in terms of the polynomials for the Weil-Petersson volumes, and is related to the invariant measure for the earthquake flow (§5).

- We conclude with equidistribution results for the level sets of the lengths of simple closed curves in moduli space(§6). Our results on the dynamics of the earthquake flow and level sets parallel known results regarding horocycle flows on homogeneous spaces [Rat2].

Now we turn to a more detailed account of our research projects. We will collect our plans for future work at the end of each section.

2 Volumes of Moduli spaces of curves

Let $S_{g,n}$ be a compact, connected, oriented surface of genus g with n boundary components $\{\beta_i\}_{i=1}^n$ with $\chi(S_{g,n}) < 0$. The mapping class group $\mathrm{Mod}_{g,n}$ of $S_{g,n}$ acts on the Teichmüller space $\mathcal{T}_{g,n}$ of complete hyperbolic Riemann surfaces marked by $S_{g,n}$. The quotient space

$$\mathcal{M}_{g,n} = \mathcal{T}_{g,n} / \mathrm{Mod}_{g,n}$$

is the moduli space of hyperbolic Riemann surfaces of genus g with n cusps. The space $\mathcal{T}_{g,n}$ is a finite-dimensional complex manifold equipped with the Weil-Petersson Kähler metric. The Weil-Petersson volume of the moduli space $\mathcal{M}_{g,n}$ is a finite number and its value as a function of g and n arises naturally in different contexts (see e.g. [KMZ] and [Pen]).

We find it fruitful to consider more generally the moduli space of bordered Riemann surfaces with fixed geodesic boundary lengths. We approach the study of the volumes of these moduli spaces via the length functions of simple closed geodesics on a hyperbolic surface.

Notation. Let

$$\mathcal{M}_{g,n}(\ell_1, \ldots, \ell_n)$$

be the moduli space of hyperbolic Riemann surfaces of genus g with n geodesic boundary components of length $\ell_1, \ldots, \ell_n$.

In [Mirz2], we establish:

Theorem 2.1 *The volume $V_{g,n}(\ell_1, \ldots, \ell_n) = \mathrm{Vol}(\mathcal{M}_{g,n}(\ell_1, \ldots, \ell_n))$ is a polynomial in $\ell_1, \ldots, \ell_n$, namely:*

$$V_{g,n}(\ell) = \sum_{|\alpha| \leq 3g-3+n} C_\alpha \cdot \ell^{\,2\alpha},$$

where $C_\alpha > 0$ lies in $\pi^{6g-6+2n-2|\alpha|} \cdot \mathbb{Q}$.

Here the exponent $\alpha = (\alpha_1, \ldots, \alpha_n)$ ranges over elements in $(\mathbb{Z}_+)^n$, $\ell^\alpha = \ell_1^{\alpha_1} \cdots \ell_n^{\alpha_n}$, and $|\alpha| = \sum \alpha_i$.

We also give an explicit recursive method for calculating these polynomials. The constant term of the polynomial $V_{g,n}(\ell)$ is the volume of $\mathcal{M}_{g,n}$, the traditional moduli space of closed surfaces of genus g with n marked points.
Example: Using our recursive method, we get:

$$V_{1,1}(\ell_1) = \ell_1^2/24 + \pi^2/6,$$

$$V_{1,2}(\ell_1, \ell_2) = (4\pi^2 + \ell_1^2 + \ell_2^2)(12\pi^2 + \ell_1^2 + \ell_2^2)/384,$$

and

$$V_{0,4}(\ell_1, \dots, \ell_4) = (4\pi^2 + \ell_1^2 + \ell_2^2 + \ell_3^2 + \ell_4^2)/4.$$

McShane identity. Our point of departure for calculating these volume polynomials is the following result [McS]:

Theorem 2.2 (McShane) *Let X be a hyperbolic once-punctured torus. Then we have*

$$\sum_{\gamma} (1 + e^{\frac{\ell_\gamma(X)}{2}})^{-1} = \frac{1}{2}, \tag{2.1}$$

where the sum is over all simple closed geodesics γ on X.

We generalize this formula to arbitrary hyperbolic surfaces with geodesic boundary components, and develop a method to integrate the generalized identity over certain coverings of $\mathcal{M}_{g,n}(\ell_1, \dots, \ell_n)$. As a result, we obtain a recursive formula for the $V_{g,n}(\ell)$'s without having to find a fundamental domain for the action of the mapping class group on the Teichmüller space [Mirz2].
Volume of the thin part of moduli space. Let $\mathcal{M}^{\epsilon}_{g,n} \subset \mathcal{M}_{g,n}$ be the set of hyperbolic Riemann surfaces whose shortest closed geodesics are of length $\leq \epsilon$. As an application of our volume calculation, we also find a formula for the Weil-Petersson volume of $\mathcal{M}^{\epsilon}_{g,n}$ for sufficiently small $\epsilon > 0$ [Mirz2].

Future research directions. One challenge is to find a McShane-type formula for closed surfaces.

We hope to explore the relation between the results above and the Selberg trace formula, by defining a zeta function related to the lengths of *simple* closed geodesics similar to the Selberg zeta function.

This analogy also motivates the study of the *simple* length spectrum of a hyperbolic surface. It is interesting to know if there is a bound on the multiplicities (the number of simple closed geodesics of the same length) depending only on g and n. The special case of this question for $g = n = 1$ is related to the uniqueness conjecture for Markoff triples [S].

3 The Kontsevich-Witten formula

By applying the method of symplectic reduction, we obtain a formula for the volume polynomial $V_{g,n}(\ell)$ in terms of the intersection numbers of tautological line bundles over $\overline{\mathcal{M}}_{g,n}$.

Theorem 3.1 *The coefficients of the volume polynomial* $\mathrm{Vol}(\mathcal{M}_{g,n}(\ell_1,\ldots,\ell_n)) = \sum C_\alpha \ell^\alpha$ *are given by*

$$C_\alpha = \frac{2^{|\alpha|}}{\alpha!} \int\limits_{\overline{\mathcal{M}}_{g,n}} \psi_1^{\alpha_1}\cdots\psi_n^{\alpha_n}\cdot w^{3g-3-|\alpha|},$$

where ψ_i *is the first Chern class of the* i*th tautological line bundle,* w *is the Weil-Petersson symplectic form,* $\alpha! = \prod \alpha_i!$ *and* $|\alpha| = \sum \alpha_i$.

Thus our algorithm for calculating volumes leads to a recursive formula for these intersection numbers leading to the Witten-Kontsevich formula [K].

More precisely, let

$$\langle\, \tau_{d_1},\ldots,\tau_{d_n}\rangle_g = \int\limits_{\overline{\mathcal{M}}_{g,n}} \prod_i \psi_i^{d_i} \quad ,$$

$$F_g(t_0,t_1,\ldots) = \sum \langle \prod \tau_{d_i}\rangle_g \prod_{r>0} t_r^{n_r}/n_r! \quad ,$$

where $n_r = \mathrm{Card}(i : d_i = r)$ and $\sum d_i = 3g-3+n$. Let

$$F = \sum_{g=0}^{\infty} \lambda^{2g-2} F_g,$$

and

$$L_n = -\left(\frac{(2n+3)!!}{2^{n+1}}\right)\frac{\partial}{\partial t_{n+1}} + \sum_{i=0}^{\infty}\left(\frac{(2i+2n+1)!!}{(2i-1)!!2^{n+1}}\right) t_i \frac{\partial}{\partial t_{i+n}}$$
$$+\frac{\lambda^2}{2}\sum_{i=0}^{n-1}\left(\frac{(2i+1)!!(2n-2i-1)!!}{2^{n+1}}\right)\frac{\partial^2}{\partial t_i \partial t_{n-1-i}},$$

where $(2i+1)!! = 1\cdot 3\ldots\cdot(2i+1)$.

Then we obtain a new proof of:

Theorem 3.2 (Witten-Kontsevich) *For* $n \geq -1$, *we have*

$$L_n(\exp(F)) = 0.$$

Analogies with moduli spaces of stable bundles. The discussion above suggests some similarities between $\mathcal{M}_{g,n}$ and the variety $\mathrm{Hom}(\pi_1(S),G)/G$ of representations of the fundamental group of the surface S in a compact Lie group G, up to conjugacy. This space is naturally equipped with a symplectic structure [Gol1]. For $G = \mathrm{SU}(2)$, the representation variety is identified with the moduli space of semi-stable holomorphic rank 2 vector bundles over a fixed Riemann surface.

For $\theta_1,\ldots,\theta_n \in G$ let

$$R_{g,n}(\theta_1,\ldots,\theta_n)$$

be the variety of representations of $\pi_1(S_{g,n})$ in $SU(2)$ such that the monodromy around β_i lies in the conjugacy class of θ_i. Here fixing the conjugacy class of the monodromy around a boundary component β corresponds to fixing the length of β in the case of $\mathcal{M}_{g,n}(\ell)$.

Like our argument for proving Theorem 3.2, it is possible to derive recursive formulas for intersection numbers of line bundles on $R_{g,n}$ by relating these numbers to the symplectic volume of $R_{g,n}(\theta_1, \dots, \theta_n)$ [Weit] and [Wit].

A surprising difference is that the action of the mapping class does not enter in the $R_{g,n}$ case. The space $R_{g,n}$ is analogous to Teichmüller space, but it has finite volume. Also, the action of the mapping class group on $R_{g,n}(\theta)$ is ergodic [Gol2].

Future research directions. It would be interesting to understand the intersection numbers involving the Chern classes of the Hodge bundle based on analogies with representation varieties as in [Je],[JeW] .

The moduli space of curves can be generalized in two ways: to the moduli space of stable maps in to a projective variety, and to the moduli space of curves with spin structures [JKV]. As in the case of $\mathcal{M}_{g,n}$, it is possible to define intersection numbers of certain natural line bundles over these spaces. An important problem is to generalize Theorem 3.2 and prove that these intersection pairings satisfy the Virasoro constraints [Pand].

We would like to generalize Verlinde type formulas for $\mathcal{L}$, the positive line bundle determined by the Weil-Petersson symplectic form.

4 Growth of the number of simple closed geodesics

We now return to the problem of the growth of $s_X(L)$, the number of simple closed curves of length $\leq L$ on X. In this section, we discuss the proof of the asymptotic formula $s_X(L) \sim n_X L^{6g-6+2n}$ based on our results on the volume of $\mathcal{M}_{g,n}(\ell)$.

For $X \in \mathcal{M}_{g,n}$, let $c_X(L)$ be the number of primitive closed geodesics on X of length $\leq L$. By work of Delsart, Huber, Selberg and Margulis, we have

$$c_X(L) \sim e^L/L$$

as $L \to \infty$. However, very few closed geodesics are *simple* [BS] and it is hard to discern them in $\pi_1(S_{g,n})$.

Notation. Let $\mathcal{ML}_{g,n}$ be the space of *measured laminations* on $S_{g,n}$. For any two isotopy classes of essential simple closed curves on $S_{g,n}$ the intersection number $i(\alpha, \beta)$ is the minimum number of points in which transverse representatives of α and β must meet. The intersection pairing extends to a continuous map $i : \mathcal{ML}_{g,n} \times \mathcal{ML}_{g,n} \to \mathbb{R}$.

There is a one-to-one correspondence between the integral measured laminations, $\mathcal{ML}_{g,n}(\mathbb{Z})$, and unions of disjoint essential simple closed curves on $S_{g,n}$, up to isotopy. There is a natural symplectic form on $\mathcal{ML}_{g,n}$ preserved by the action of $\mathrm{Mod}_{g,n}$.

For any $X \in \mathcal{T}_{g,n}$ and $\lambda \in \mathcal{ML}_{g,n}$, let $\ell_\lambda(X)$ denote the hyperbolic length of λ on X.

Counting problems. To understand the growth of $s_X(L)$, it proves fruitful to fix a simple closed curve $\gamma \in \mathcal{ML}_{g,n}(\mathbb{Z})$ and consider more generally the counting function

$$s_X(L,\gamma) = \#\{\alpha \in \mathrm{Mod}_{g,n} \cdot \gamma \mid \ell_\alpha(X) \le L\}.$$

There are only finitely many isotopy classes of simple closed curves on $S_{g,n}$ up to the action of the mapping class group. Therefore, summing $s_X(L,\gamma)$ over representatives of these orbits gives $s_X(L)$, and the asymptotics of the $s_X(L,\gamma)$'s determines the asymptotics of $s_X(L)$.

In [Mirz1] we show :

Theorem 4.1 *For any $\gamma \in \mathcal{ML}_{g,n}(\mathbb{Z})$, we have*

$$\lim_{L\to\infty} \frac{s_X(L,\gamma)}{L^{6g-6+2n}} = n_\gamma(X),$$

where $n_\gamma(X)$ is a smooth proper function of $X \in \mathcal{M}_{g,n}$.

In the case of $\mathcal{M}_{1,1}$, this result was previously obtained by McShane and Rivin [MR]. The upper and lower estimates for $S_X(L)$ when $X \in \mathcal{M}_{g,n}$ were obtained by M. Rees in [Rs] and I. Rivin in [Ri].

Idea of the proof. The crux of matter is to understand the density of $\mathrm{Mod}_{g,n} \cdot \gamma$ in $\mathcal{ML}_{g,n}(\mathbb{Z})$. This is similar to the problem of the density of relatively prime pairs (p,q) in $\mathbb{Z}^2$. Our approach is to use the moduli space $\mathcal{M}_{g,n}$ to understand the average of these densities. Appealing to Theorem 2.1, we show that the average defined by

$$S(L,\gamma) = \int_{\mathcal{M}_{g,n}} s_X(L,\gamma)\, dX$$

is well-behaved; in fact it is a polynomial in L. Here the integral on $\mathcal{M}_{g,n}$ is taken with respect to the Weil-Petersson volume form. This polynomial behaviour allows us to use the ergodicity of the action of the mapping class group on $\mathcal{ML}_{g,n}$ [Mas2] to prove that these densities exist. Then Theorem 4.1 follows by a simple lattice-counting argument.

Frequencies of different types of simple closed curves. We now discuss more precisely how $n_\gamma(X)$, the constant in the growth rate of $s_X(L,\gamma)$, depends on X and on the simple closed curve γ.

Let B_X be the unit ball in the space of measured geodeic laminations with respect to the length function at X:

$$B_X = \{\lambda \mid \ell_\lambda(X) \le 1\} \subset \mathcal{ML}_{g,n}.$$

We show that B_X is convex with respect to the piecewise linear structure of $\mathcal{ML}_{g,n}$. Let $B(X) = \mathrm{Vol}(B_X)$ with respect to the Thurston volume form on $\mathcal{ML}_{g,n}$. We show that

$$b_{g,n} = \int_{\mathcal{M}_{g,n}} B(X)\, dX$$

is a finite number in $\pi^{6g-6+2n} \cdot \mathbb{Q}$ which can be calculated in terms of the leading coefficients of the volume polynomials.

We show that the contributions of X and γ to $n_\gamma(X)$ separate as follows:

Theorem 4.2 *For any $\gamma \in \mathcal{ML}_{g,n}(\mathbb{Z})$, there exists a rational number c_γ such that we have:*

$$n_\gamma(X) = \frac{c_\gamma \cdot B(X)}{b_{g,n}}.$$

It follows that the relative frequencies of different types of simple closed curves on X are universal rational numbers.

Corollary 4.3 *For $X \in \mathcal{M}_{g,n}$ and $\gamma_1, \gamma_2 \in \mathcal{ML}_{g,n}(\mathbb{Z})$, we have*

$$\lim_{L\to\infty} \frac{s_X(L,\gamma_1)}{s_X(L,\gamma_2)} = \frac{c_{\gamma_1}}{c_{\gamma_2}} \in \mathbb{Q}.$$

The limit is a positive rational number independent of X.

Remark. The same result holds when the surface X has variable negative curvature.

Example: Let γ_i be a simple closed curve on $S_{g,0}$ such that $S_{g,0} - \gamma_i \cong S_{i,1} \cup S_{g-i,1}$. Then we have

$$\frac{s_X(L,\gamma_i)}{s_X(L,\gamma_j)} \to \frac{\binom{g}{i}}{\binom{g}{j}}$$

as $L \to \infty$.

The frequency $c_\gamma \in \mathbb{Q}$ of a given simple closed curve can be described in a purely topological way as follows:

Theorem 4.4 *For any $\gamma \in \mathcal{ML}_{g,n}(\mathbb{Z})$, we have*

$$\frac{\#(\{\lambda \in \mathcal{ML}_{g,n}(\mathbb{Z}) \mid i(\lambda,\gamma) \le k\}/\operatorname{Stab}(\gamma))}{k^{6g-6+2n}} \to c_\gamma$$

as $k \to \infty$.

Note that $c_\gamma = c_\delta$ for all $\delta \in \mathrm{Mod}_{g,n} \cdot \gamma$.

We can also calculate c_γ recursively using our recursive formula for $V_{g,n}(\ell)$. In fact, we can write the number c_γ in terms of the intersection numbers of tautological line bundles over the moduli space of Riemann surfaces of type $S_{g,n} - \gamma$. This is analogous to the situation for counting branched coverings of $\mathbb{P}^1$ (Hurwitz numbers); these numbers also can be expressed in terms of tautological intersection products in $\overline{\mathcal{M}}_{g,n}$[PO].

Future research directions. We would like to find a different proof of Corollary 4.3 using loop-erased random walks or combinatorial methods.

It would be interesting to see if and the Euler characteristic of $\mathcal{M}_{g,n}$, calculated in [HZ], can be related to $b_{g,n}$ via a Gauss-Bonnet type theorem.

We would like to strengthen the asymptotic formula for $s_X(L,\gamma)$ to include an error term of the form

$$s_X(L,\gamma) \sim n_X(\gamma) L^{6g-6+2n} + O(L^{\alpha})$$

as $L \to \infty$, for some $\alpha < 6g-6+2n$.

Similar counting problems for the number of saddle connections for a *generic* Abelian differential have been studies by Masur, Eskin and Zorich [EMZ]. Constants in the quadratic asymptotics and frequencies of different types of saddle connections are related to volumes of moduli spaces of holomorphic Abelian differentials. We would like to find a different way to calculate these constants by using ergodic theory and the symplectic structure of moduli spaces of holomorphic Abelian differentials.

5 From $\mathcal{ML}_{g,n}$ to holomorphic quadratic differentials

In this section, we discuss the relationship between the earthquake flow on $\mathcal{PM}_{g,n}$ the bundle of geodesic measured laminations and the Teichmuller horocycle flow on $\mathcal{QM}_{g,n}$ the bundle of holomorphic quadratic differentials. Our main result is:

Theorem 5.1 *The earthquake flow and the Teichmüller horocycle flow are measurably isomorphic.*

It is known that the Teichmüller horocycle flow is ergodic with respect to the Lebesgue measure class [Mas2]. Therefore, we have:

Corollary 5.2 *The earthquake flow on $\mathcal{P}^1\mathcal{M}_{g,n}$ is ergodic with respect to the Lebesgue measure class.*

We also discuss the invariant measure for the earthquake flow on $\mathcal{P}^1\mathcal{M}_{g,n}$. Then by using the relation between $\mathcal{P}^1\mathcal{M}_{g,n}$ and $\mathcal{Q}^1\mathcal{M}_{g,n}$, we obtain:

Theorem 5.3 *We have:*

$$\mathrm{Vol}(\mathcal{Q}^1\mathcal{M}_{g,n}) = \int\limits_{\mathcal{M}_{g,n}} B(X)\, dX.$$

The value of $\mathrm{Vol}(\mathcal{Q}^1\mathcal{M}_{g,n})$ arises in several problems related to billiards and dynamics of interval exchange maps. Volumes of different strata of moduli spaces of holomorphic Abelian differentials have been calculated in [EO], but the case of $\mathcal{Q}^1\mathcal{M}_{g,n}$ seems to be new.

Notation. Attached to $S_{g,n}$ one has:

- $\mathcal{QT}_{g,n} \to \mathcal{T}_{g,n}$, the bundle of holomorphic quadratic differentials;

- $\mathcal{Q}^1\mathcal{T}_{g,n}$, the unit sub-bundle for the norm

$$\| q \| = \int_X |q|;$$

- $\mathcal{QM}_{g,n} = \mathcal{QT}_{g,n}/\operatorname{Mod}_{g,n}$, the moduli space of holomorphic quadratic differentials;
- $\mathcal{PT}_{g,n} = \mathcal{ML}_{g,n} \times \mathcal{T}_{g,n}$, the bundle of geodesic measured laminations over $\mathcal{T}_{g,n}$;
- $\mathcal{P}^1\mathcal{T}_{g,n}$, the unit sub-bundle for the norm

$$\| (\lambda, X) \| = \ell_\lambda(X);$$

  and finally,
- $\mathcal{PM}_{g,n} = \mathcal{PT}_{g,n}/\operatorname{Mod}_{g,n}$.

We wish to compare the dynamics of Thurston's earthquake flow on $\mathcal{Q}^1\mathcal{M}_{g,n}$ to the Teichmuller horocycle flow on $\mathcal{P}^1\mathcal{M}_{g,n}$. These flows are defined as follows.

- Thurston's earthquake flow on $\mathcal{PT}_{g,n}$ is defined at time t by

$$(X, \lambda) \xrightarrow{t} (Tw_{t\lambda}(X), \lambda),$$

  where for a simple closed curve γ on $X \in \mathcal{T}_{g,n}$, $Tw_{t\gamma}(X) \in \mathcal{T}_{g,n}$ is constructed by cutting X along γ, twisting distance t to the right, and regluing [Th] .
- Any holomorphic quadratic differential $q \in \mathcal{QT}_{g,n}$ can be defined with a pair of transverse measured foliations on $S_{g,n}$: the horizontal and vertical foliations of q. In local coordinate charts (x, y) on $S_{g,n}$ in which $q = (dx + i\ dy)^2$ the vertical measured foliation of $h^t(q)$ is determined by the from $|dx + t\ dy|$; that is h^t acts by the matrix

$$\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

  on $\mathcal{QT}_{g,n}$ [Mas1].

Invariant measures for the earthquake flow. It is known that there exists a finite invariant measure $\mu_{g,n}$ for the horocycle flow on $\mathcal{Q}^1\mathcal{M}_{g,n}$ [Mas1]. We prove:

Theorem 5.4 *There exists a finite invariant measure $\nu_{g,n}$ for the earthquake flow on $\mathcal{P}^1\mathcal{M}_{g,n}$. This measure projects to the volume form given by $B(X) \cdot \mu_{wp}$ on $\mathcal{M}_{g,n}$.*

Any point $X \in \mathcal{T}_{g,n}$ define a measure ν_X on $\mathcal{PML}_{g,n}$ such that for $A \subset \mathcal{PML}_{g,n}$, we have

$$\nu_X(A) = \mathrm{Vol}(\{\lambda \,|[\lambda] \in A,\ \ell_\lambda(X) \le 1\}),$$

with respect to the Thurston volume form on $\mathcal{ML}_{g,n}$. The Weil-Petersson volume form on $\mathcal{M}_{g,n}$ and the family $\{\nu_X\}_{X \in \mathcal{T}_{g,n}}$ of measures on $\mathcal{PML}_{g,n}$ combine to give the invariant measure $\nu_{g,n}$.

Since $\nu_X(\mathcal{PML}_{g,n}) = B(X)$, the results of §4 imply

$$\mathrm{Vol}(\mathcal{P}^1\mathcal{M}_{g,n}) = \int\limits_{\mathcal{M}_{g,n}} B(X)\, dX < \infty.$$

Idea of the proof of Theorem 5.1**.** Using work of Bonahon [Bon] and Thurston [Th] on shearing and cataclysm coordinates for the Teichmüller space, we show that:

Theorem 5.5 *For any complete measured lamination λ, there exists a symplectomorphism*

$$F_\lambda : \mathcal{T}_{g,n} \to \mathcal{ML}_{g,n}$$

such that $\ell_\lambda(X) = i(\lambda, F_\lambda(X))$.

The earthquake and Teichmuller horocycle flow are the Hamiltonian flows for the length and intersection function [Pap]. Since F_λ sends one function to the other, we have:

Corollary 5.6 *The map F_λ sends the earthquake flow for λ to the corresponding Teichmüller horocycle flow in a time-preserving fashion.*

As the map F_λ is defined for almost every $\lambda \in \mathcal{ML}_{g,n}$, the measurable map $F : \mathcal{PT}_{g,n} \to \mathcal{QT}_{g,n}$, defined by

$$F(\lambda, X) = (\lambda, F_\lambda(X))$$

sends the earthquake flow to the corresponding Teichmüller horocycle flow. It is clear that F descends to a map

$$F : \mathcal{P}^1\mathcal{M}_{g,n} \to \mathcal{Q}^1\mathcal{M}_{g,n},$$

which will be denoted by the same letter. By Theorem 5.5, we have

$$F^*(\mu_{g,n}) = \nu_{g,n},$$

and we obtain Theorem 5.1 and Theorem 5.3.

Analogy with negatively curved spaces. For each $X \in \mathcal{T}_{g,n}$, the measure ν_X on $\mathcal{PML}_{g,n}$ is analogous to the visual measure on S^{n-1}_∞ seen from $x \in \mathbb{H}^n$. For $X, Y \in \mathcal{T}_{g,n}$, define $G_{X,Y} : \mathcal{PML}_{g,n} \to \mathbb{R}_+$ by

$$G_{X,Y}([\lambda]) = \log\left(\frac{\ell_\lambda(X)}{\ell_\lambda(Y)}\right).$$

Then we have

$$\frac{d\nu_X}{d\nu_Y}[\lambda] = e^{G_{X,Y}[\lambda]}.$$

It would be interesting to understand this analogy better.

6 Equidistribution results and counting simple closed geodesics

The growth of the number of simple closed geodesics, $s_X(L)$, can be also investigated via the dynamics of Thurston's earthquake flow on moduli space. In this section, we present our results on equidistribution of *horospheres* and obtain another proof of the asymptotic formula $s_X(L) \sim n_X L^{6g-6+2n}$ as $L \to \infty$.

Our equidistribution results suggest more analogies between the earthquake flow and unipotent flows on homogeneous spaces. The latter are now rather well-understood by work of Ratner, Margulis and Dani [Rat2].

Horosphere measures. For any $\gamma \in \mathcal{ML}_{g,n}(\mathbb{Z})$ and $L > 0$, we define an ergodic, earthquake-flow invariant probability measure $\nu_\gamma(L)$ on $\mathcal{PM}_{g,n}$ supported on the set

$$H_\gamma(L) = \{(X,\lambda) \mid \ell_\gamma(X) = L,\ \ell_\lambda(X) = 1,\ i(\gamma,\lambda) = 0\} / \operatorname{Mod}_{g,n} \subset \mathcal{P}^1\mathcal{M}_{g,n}.$$

The preimage of $H_\gamma(L)$ in $\mathcal{PT}_{g,n}$ lies above the set

$$\{X \mid \ell_\gamma(X) = L\ \} \subset \mathcal{T}_{g,n}$$

which is analogous to a *horosphere*.

By using the ergodicity and nondivergence of the earthquake flow [MW], we prove the following equidistribution result:

Theorem 6.1 *As L tends to infinity, the horosphere meaure $v_\gamma(L)$ become equidistributed with respect to the unique invariant probability measure in the Lebesgue measure class; that is,*

$$\nu_\gamma(L) \to \nu_{g,n}/b_{g,n}$$

as $L \to \infty$.

In particular, if γ is a maximal system of simple closed curves then by using the fact that $v_X(\mathcal{PML}_{g,n}) = B(X)$, we show that the images of the level sets

$$\{X \in \mathcal{T}_{g,n} \mid \ell_\gamma(X) = L\}$$

become equidistributed with respect to the measure $B(X) \cdot \mu_{wp}$ in $\mathcal{M}_{g,n}$ as $L \to \infty$.

Counting simple closed geodesics. By studying the asymptotic behavior of the family of invariant measures mentioned above, we obtain a stronger version of Theorem 4.1:

Corollary 6.2 *For $X \in \mathcal{M}_{g,n}$, we have*

$$\frac{s_X(L+1,\gamma) - s_X(L,\gamma)}{L^{6g-6+2n-1}} \to n_\gamma(X)$$

as $L \to \infty$.

Earthquake balls. We show that $B(X)$ is related to the growth rate of the volume of the earthquake ball.

Theorem 6.3 *Let $B_X(r)$ be the Weil-Petersson volume of the earthquake ball about $X \in \mathcal{T}_{g,n}$ of radius r; that is*

$$B_X(r) = \mathrm{Vol}(\{Tw_{t\ \lambda}(X) \mid T \leq r, \ell_\lambda(X) \leq 1\}).$$

Then we have

$$\frac{B_X(r)}{r^{6g-6+2n}} \to B(X)$$

as $r \to \infty$.

Analogy with counting integral points on affine varieties. Let $N(V,L)$ be the number integral points $x \in V(\mathbb{Z})$ of $\| x \| \leq L$ on the affine homogeneous variety $V = G(\mathbb{Z})\backslash G/H$. There are similarities between the problem of the growth of $s_X(L)$ and the problem of the growth of $N(V,L)$ as $L \to \infty$. Understanding the asymptotics of $N(V,L)$ leads to a proof of Siegel's mass formula [EsM], [ERS].

In this analogy we have the following correspondences:

$$V \longleftrightarrow \mathcal{ML}_{g,n}$$
$$V(\mathbb{Z}) \longleftrightarrow \mathcal{ML}_{g,n}(\mathbb{Z})$$
$$G(\mathbb{Z}) \longleftrightarrow \mathrm{Mod}_{g,n}$$
$$\| \, . \, \| \longleftrightarrow \ell(X)$$

In both cases one reduces the counting problems to the case of a single orbit, and uses equidistribution results to obtain asymptotics.

Future research directions. We would like to classify the ergodic measures for the earthquake flow and prove results analogous to Ratner's rigidity theorems for unipotent flows on homogeneous spaces [Rat1]. We speculate that any ergodic earthquake flow invariant measure is "geometric". Understanding the closure of orbits of the earthquake flow in $\mathcal{P}^1\mathcal{M}_{g,n}$ would shed light to the classification of ergodic measures for $PSL(2,\mathbb{R})$ action on the moduli space of holomorphic quadratic differentials and characterizing Veech curves (as in [Mc2]). Also, in case when λ is a pseudo-Anosov lamination, this would be related to the topology of the complexifaction of closed Teichmüller geodesics in $\mathcal{M}_{g,n}$ as in [Mc3].

A more approachable problem is the classification of the ergodic measures of the action of the mapping class group on $\mathcal{ML}_{g,n}$.

The problem of classification of the ergodic measures of the earthquake flow might be useful for problems related to the Kazhdan Property T of the mapping class group [GW].

It is an interesting open problem to know if the earthquake flow and horocycle flow are actually topologically equivalent. Also, we do not know if the earthquake flow can be extended to a $PSL(2, \mathbb{R})$ action on $\mathcal{PM}_{g,n}$.

The isomorphism in Theorem 5.5 suggests that the stretch path (defined in [Th]) plays the role of the geodesic flow for the earthquake flow. We plan to study basic properties of these paths. In particular, it is important to study recurrent stretch paths.

The map F_λ (as in Theorem 5.5) is only defined for complete maximal measured laminations. A project that we are currently working on is to extend this map for non-complete measured laminations. We would like to relate the image of this map to the stratum $\mathcal{Q}_{g,n}(a_1, \ldots, a_k)$, of the moduli space of holomorphic quadratic differentials, that is the space of pairs (X, ϕ) where $X \in \mathcal{M}_{g,n}$ and ϕ is a holomorphic quadratic differential on X with zeros of order $a_1, \ldots, a_k$. We can show that the geometric structure of the complementary regions determines the type of the quadratic differential. We would like to generalize the existing construction by using special train tracks and obtain a counting result analogous to Corollary 4.3 in this setting. This might be useful for counting problems related to interval exchange maps [Zo]. It also would yield information about the volume of $\mathcal{Q}^1_{g,n}(a_1, \ldots, a_n)$ as defined in [Mas1].

The problem of the classification of all ergodic measures for the earthquake or horocycle flow would shed light on the asymptotic behaviour of the number of saddle connections of *all* 1-forms (not just generic ones).

There is a one-to-one correspondence between simple closed curves and closed orbits for the earthquake flow. It would be interesting to define a dynamical zeta function related to the periodic trajectories of the earthquake flow similar to the dynamical zeta functions that can be defined for Axiom A flows [PP].

7 Additional research directions

We conclude by mentioning some other directions for future research that interest us.

$\mathrm{PSL}(2, \mathbb{C})$**-character variety and quadratic differentials.** Moduli spaces of holomorphic quadratic differentials are also closely related to the representation variety

$$V(S) = \mathrm{Hom}^{irr}(\pi_1(S), PSL(2, C))/PSL(2, C),$$

and $P(S)$, the moduli space of complex projective structures on S. By work of Kawai [Ka] the complex-symplectic structure on $P(S)$ from the holomorphic quadratic differentials equals the complex symplectic structure induced by the holonomy map from $\mathrm{PSL}(2, \mathbb{C})$-character variety. Also the complex twist vector field (related to grafting in [Mc1]) is Hamiltonian with respect to the complex length function. We are interested in understanding the natural line bundles induced on $P(S)$ as a representation variety (especially comparing them to classes discussed in [Ko]), and also the interpretation of the geometric data like non-realizable laminations [Bow] in holomorphic the quadratic differentials setting.

Also, we are interested in calculating the volume of the moduli space of quadratic differentials with fixed holonomy around boundary components. This would give rise to a different proof of the counting results obtained in [EMZ].

Topological dynamics on representation variety. In case of the representation variety to $SU(2)$, it is known [PX] that if $\sigma \in \mathrm{Hom}\big(\pi_1(\Sigma), \mathrm{SU}(2)\big)$ is such that $\sigma\big(\pi_1(\Sigma)\big)$ is dense in SU(2) then the $\mathrm{Mod}_{g,n}$-orbit of the conjugacy class $[\sigma]$ is dense in the representation variety. We would like to prove analogous results in the $PSL(2,\mathbb{C})$ case.

Dynamics on rational surfaces. We are interested in studying the dynamics of automorphisms of compact complex manifolds. If the entropy of an automorphism of X, a Kähler surface, is zero then the dynamics is completely understood. One is led to consider automorphisms of positive entropy [Ca]. A surface with a positive entropy automorphism is a torus, a $K3$ surface, an Enrique surface or a non-minimal rational surface. In a joint project with Izzet Coskun, we study positive entropy automorphisms of non-minimal rational surfaces by investigating the set of periodic points and curves.

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